ACTION OF SHARP DIES ON A CYLINDRICAL SHELL OF INFINITE LENGTH

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The framework of the theory of thin elastic shells is used to obtain the solution of the problem of impressing identical sharp dies into a shell along circumferential arcs. The edges of the dies are assumed to be perfectly rigid, are of constant radius and have no corners. The character of the reactions of the dies and the size of the zone of contact is established.

It is shown that despite the absence of sharp corners the reactions of the dies increase without bounds near the edges of the zone of contact and have a root-type singularity.

The solution of the problem is constructed with the help of the theory of singular integral equations and is reduced to a linear system of algebraic equations.

1. Consider a thin elastic cylindrical shell of infinite length (Fig. 1) compressed by m equal sharp dies (the figure illustrates the case $m = 2$). The curvature of the sharp edge of the die is assumed constant and equal to $1 / R_1$. The general case when the edge curvature is variable is treated in the analogous manner. The die edge is assumed to be perfectly rigid. An external force P / m applied to each die produces a zone of contact characterized by the angle θ (Fig. 1.) Neglecting the friction between the shell and the die we can reduce the problem to that of determining the normal reaction q, acting from the direction of the die on the shell and the magnitude of the zone of contact θ. The linear theory of shells presupposes that either the angle θ is small, or that the radius of the die edge differs little from the radius R of the shell.

Assuming that within the zone of contact the shell (of thickness h) is in close contact with the die, we obtain the initial equation of the problem by assuming the bending deformation of the shell under the die to be equal to $1 / R_1 - 1 / R$.

Using the results of [1] we can write the bending deformation of the shell on the line of contac $-\theta < \varphi < \theta$ in the form

$$\varkappa_2(\alpha) = \frac{1-\nu^2}{4\pi E h R a^2 m} \int_{-\beta}^{\beta} \left[\ln \left| 2 \sin \frac{\alpha - \alpha_0}{2} \right| - K(\alpha - \alpha_0)\right] q \, d\alpha_0 \tag{1.1}$$

$$\alpha = m\varphi, \quad \alpha_0 = m\varphi_0, \quad \beta = m\theta, \quad a^2 = h^2 / 12R^2$$

$$K(\alpha - \alpha_0) = \sum_{k=1}^{\infty} \frac{b_k}{k} \cos k(\alpha - \alpha_0) \tag{1.2}$$

$$b_k = -1 + \frac{1}{\Delta} \sum_{j=1}^{2} (a_j c_j + b_j d_j), \qquad n = km \tag{1.3}$$

$$a_j = \frac{B_j \mp 4Aq_j^2}{q_j (p_j^2 + q_j^2)} \qquad b_i = \frac{B_j \pm 4Ap_j^2}{p_j (p_j^2 + q_j^2)}, \qquad p_j + iq_j = \mu_j$$

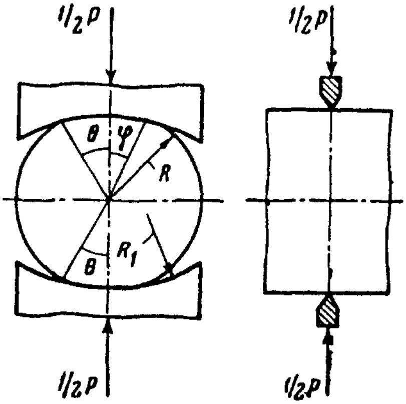

Fig. 1.

$$B_j = A^2 \mp 4\,(p_1^2 q_1^2 - p_2^2 q_2^2)$$

$$A = p_1^2 - q_1^2 - p_2^2 + q_2^2$$

$$\Delta = [A^2 + 4\,(p_1 q_1 + p_2 q_2)^2]\,[A^2 + 4\,(p_1 q_1 - p_2 q_2)^2]$$

where μ_j are the roots of the characteristic equation [1], the upper sign corresponds to $j = 1$, and the lower sign to $j = 2$.

For a shallow shell we have

$$c_j + id_j = (\mu_j^2 + 1)^2 - \frac{2+\nu}{n^2}\mu_j^2 - \frac{1}{n^2}$$

The first term in (1.1) represents the principal part of the Green function for the deformation $\varkappa_2$ and the quantity $K(\alpha - \alpha_0)$ is a regular kernel.

The initial equation of the problem will, in accordance with the previous statement, have the form

$$\int_{-\beta}^{\beta} \ln\left|2\sin\frac{\alpha-\alpha_0}{2}\right| q\,d\alpha_0 = \int_{-\beta}^{\beta} K(\alpha - \alpha_1)\,q\,d\alpha_1 - \omega_0 m \tag{1.4}$$

$$\omega_0 = \frac{4\pi E h a^2}{1-\nu^2}\left(1 - \frac{R}{R_1}\right) \qquad (-\beta < \alpha < \beta)$$

Its solution which will obviously be even when $\omega_0 =$ const obeys the condition

$$\int_{-\beta}^{\beta} q\cos\frac{\alpha}{m}\,d\alpha = \frac{P}{R} \tag{1.5}$$

Here $P\,/\,m$ denotes the external force applied to the die (Fig. 1). Condition (1.5) connects the angle β, characterizing the size of the zone of contact with the external force P.

It appears that the most rational method of investigating the integral equation of the first kind (1.4) would involve its transformation into a Fredholm's integral equation of the second kind. This would expose the character of the reaction and allow the use of the examples of already known solutions. We shall perform the transformation by solving (1.4) under the assumption that its right-hand side is known. An analogous procedure is adopted in regularizing singular integral equations (the method of Carleman [2]) and the process is equally applicable to the present case [3].

2. Consider the equation

$$\int_{-\beta}^{\beta} \ln\left|2\sin\frac{\alpha-\alpha_0}{2}\right| q\,d\alpha = f(\alpha_0) \tag{2.1}$$

where $f(\alpha_0)$ is a known, even function. The following is the simplest method of solution. Integrating the left-hand side by parts we reduce (2.1) to the form

$$\int_{-\beta}^{\beta} \frac{1}{2} \operatorname{ctg} \frac{\alpha-\alpha_0}{2} Q d\alpha = A_1 \ln X(\alpha_0) - f(\alpha_0) \tag{2.2}$$

$$Q = \int_0^{\alpha} q d\alpha, \qquad A_1 = 2\int_0^{\beta} q d\alpha, \qquad X(\alpha) = \sqrt{2(\cos\alpha - \cos\beta)} \tag{2.3}$$

Substitution of $t = e^{i\alpha}$ into (2.2) yields a singular integral equation with a Cauchy kernel, whose solutions are known [3]. The physical condition of integrability of q requires by virtue of (2.2), that the function Q has a bounded solution. Such solution will be unique and on returning to the previous variable α it will assume the form

$$Q(\alpha_0) = -\frac{X(\alpha_0)}{2\pi^2} \int_{-\beta}^{\beta} \frac{A_1 \ln X(\alpha) - f(\alpha)}{X(\alpha) \sin{}^1/_2(\alpha-\alpha_0)} d\alpha \tag{2.4}$$

This solution exists provided that the condition

$$\int_{-\beta}^{\beta} \frac{A_1 \ln X(\alpha) - f(\alpha)}{X(\alpha)} \cos\frac{\alpha}{2} d\alpha = 0 \tag{2.5}$$

holds. We have the following differentiation formula

$$\frac{d}{d\alpha_0}\left[X(\alpha_0) \int_{-\beta}^{\beta} \frac{f(\alpha)d\alpha}{X(\alpha)\sin{}^1/_2(\alpha-\alpha_0)}\right] = \frac{1}{X(\alpha_0)} \int_{-\beta}^{\beta} \frac{X(\alpha) f'(\alpha)}{|\sin{}^1/_2(\alpha-\alpha_0)} d\alpha \tag{2.6}$$

This formula holds for any function $f(\alpha)$ such that the function $X(\alpha) f(\alpha)$ is continuous and vanishes at the end points of the interval $(-\beta, \beta)$. The formula is proved by passing to the variable $t = \exp(i\alpha)$ in its left-hand side and performing certain transformations, as well as making use of a well known theorem according to which the limit value of the derivative of a Cauchy-type integral is equal to the derivative of its limit value.

The solution of (2.1) is obtained, in accordance with (2.3), by differentiating (2.4) and taking into account both, the formula (2.6) and the value of the integral

$$\int_{-\beta}^{\beta} X(\alpha) \frac{d \ln X(\alpha)}{d\alpha} \frac{d\alpha}{\sin{}^1/_2(\alpha-\alpha_0)} = -2\pi\cos\frac{\alpha_0}{2} \tag{2.7}$$

This solution has the form

$$q(\alpha_0) = \frac{1}{2\pi^2 X(\alpha_0)} \int_{-\beta}^{\beta} \frac{X(\alpha)}{\sin{}^1/_2(\alpha-\alpha_0)} f'(\alpha) d\alpha + \frac{A_1}{\pi} \frac{\cos{}^1/_2\alpha_0}{X(\alpha_0)} \tag{2.8}$$

The constant A_1 is obtained from the condition (2.5) which, after computing the integral of $\ln X(\alpha)$ becomes

$$A_1 \ln \sin\frac{\beta}{2} = \frac{1}{\pi} \int_{-\beta}^{\beta} \frac{f(\alpha)}{X(\alpha)} \cos\frac{\alpha}{2} d\alpha \tag{2.9}$$

Thus Eq. (2.1) has a unique solution unbounded at the end points and fully defined. An alternative method of solving (2.1) may be found e.g. in [4]. Equations of the type (2.2) belong to the class of equations with automorphic kernels [5]. A method for their solution is given in [6].

3. Inserting now Eq. (2.1) into the solution (2.8) and replacing the function $f(\alpha)$ in the right-hand side of the initial equation (1.4) with Eq. (2.9), we obtain the Fredholm integral equation of the second kind for determining the reaction q as well as an equation defining the constant A_1. Changing the order of integration in the repeated integrals, solving the integrals and taking into account the parity of the solution of q we obtain the above-mentioned equations in the form

$$q(\alpha_0) + \frac{1}{\pi X(\alpha_0)} \int_{-\beta}^{\beta} R(\alpha_1, \alpha_0) q d\alpha_1 = \frac{A_1 \cos {}^1/_2\alpha_0}{\pi X(\alpha_0)} \tag{3.1}$$

$$A_1 \ln \sin \frac{\beta}{2} = \int_{-\beta}^{\beta} \Psi(\alpha_1) q d\alpha_1 - \omega_0 m \tag{3.2}$$

$$R(\alpha_1, \alpha_0) = \sum_{k=1}^{\infty} b_k \cos k\alpha_1 J_k(\alpha_0) \tag{3.3}$$

$$\Psi(\alpha_1) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{b_k}{k} (P_k + P_{k-1}) \cos k\alpha_1 \tag{3.4}$$

Here $P_k = P_k(\cos\beta)$ are the Legendre polynomials which were computed with the aid of the following integral representation [7]

$$\int_0^{\beta} \cos\left(k + \frac{1}{2}\right)\alpha \frac{d\alpha}{X(\alpha)} = \frac{\pi}{2} P_k$$

$$J_k(\alpha_0) = \frac{1}{2\pi} \int_{-\beta}^{\beta} X(\alpha) \sin k\alpha \frac{d\alpha}{\sin {}^1/_2(\alpha - \alpha_0)} = \sum_{m=0}^{k} a_m \cos\left(k - m + \frac{1}{2}\right)\alpha_0 \tag{3.5}$$

$$a_0 = 1, \qquad a_1 = -\cos\beta, \qquad a_m = \frac{P_{m-2} - P_m}{2m - 1} \ (m = 2, 3, \ldots) \tag{3.6}$$

The integrals (3.5) were found by changing the variable to $t = e^{i\alpha}$ and applying the theorem of residues [8, p. 398]. Equation (3.1) has a unique solution which is unbounded at the end points and possesses a singularity of the type $1 / X(\alpha)$, where $X(\alpha)$ is a canonical function given by (2.3).

This solution is fully defined if the size of the zone of contact β is given. The latter is determined by (1.5). It should be noted that the concentration of the reaction at the boundary of the zone of contact is also observed when a plate is in contact with a rigid surface [9], when two shells are in contact, etc. A similar phenomenon distinguishes essentially the contact problems of the theory of elasticity where the normal reaction vanishes at the boundary of the zone of contact in the case when the die has no sharp corners. This follows from the specific properties of the equations of the theory of shells constructed in accordance with the Kirchhoff-Love hypothesis.

4. Omitting the investigation of convergence of the series (1.2), we quote the values of its coefficients (1.3) for the shells of diameter $R / h = 100$ and $m = 2$

k	1	3	5	7	9	11	13
$-b_k 10^4$	8799	4838	1801	629	258	124	68.4
k	15	17	19	21	23	25	27
$-b_k 10^4$	41.2	26.8	18.5	13.3	9.97	7.71	6.12

The coefficients b_k decrease at large values of $n = mk$ as $1 / n^4$. Terminating the series (3.3) after M terms we obtain, instead of (3.1), an equation with a degenerate kernel. Its solution has the form

$$q = \frac{A_1 y(\alpha)}{\pi X(\alpha)}, \quad y(\alpha) = \cos\frac{\alpha}{2} - \sum_{k=1}^{M} C_k b_k \sum_{s=0}^{k} a_s \cos\left(k - s + \frac{1}{2}\right)\alpha \tag{4.1}$$

Here b_k and a_s are the coefficients given by (1.3) and (3.6), C_k are the coefficients determined from the solution of the following system of algebraic linear equations

$$C_k + \sum_{n=1}^{M} \alpha_{kn} C_n = \frac{1}{2}(P_k + P_{k-1}) \qquad (k = 1, 2, \ldots, M) \tag{4.2}$$

$$\alpha_{kn} = \frac{1}{2} b_n \sum_{s=0}^{n} a_s (P_{n-s+k} + P_{n-s-k}) \tag{4.3}$$

$P_k = P_k(\cos\beta)$ are the Legendre polynomials and $P_{-n} = P_{n-1}$. The constant A_1 is obtained from (3.2) by inserting into it the expression (4.1) for q. Solving the integrals then yields

$$A_1 = -\omega_0 m \left[\ln\sin\frac{\beta}{2} - \frac{1}{2}\sum_{k=1}^{M} \frac{b_k C_k}{k}(P_k + P_{k-1})\right]^{-1} \tag{4.4}$$

The connection between the external force P and the size of the zone of contact is given by (1.5). Expanding $\cos(\alpha / m)$ into a series in $\cos n\alpha$ and inserting q from (4.1), we reduce (1.5) to the form

$$\frac{P}{R} = A_1 C_1, \qquad m = 1$$

$$\frac{P}{R} = A_1 \frac{2m}{\pi} \sin\frac{\pi}{m}\left(\frac{1}{2} - \frac{1}{2}\sum_{k=1}^{M} C_k b_k \sum_{s=0}^{k} a_s P_{k-s} + \sum_{n=1}^{\infty}\right)\frac{(-1)^{n+1} C_n}{n^2 m^2 - 1} \qquad (m > 1) \tag{4.5}$$

where C_k is the solution of (4.2). The first formula of (4.5) can also be used when $m > 1$ for small β, in which case $\cos(\alpha / m) \approx \cos\alpha$.

We note that $\beta < \pi$ always, since for $\beta = \pi$ the quantity $A_1 = \infty$ by (4.4) and this corresponds to an unlimited external force P.

Equation (3.1) can also be solved numerically using the numerical integration formulas. In this case however, a new variable x, defined by $\sin {}^1/_2\alpha = \sin {}^1/_2 \beta \sin {}^1/_2 x$ must be used. This alters the limits of integration from $(-\beta, \beta)$ to $(-\pi, \pi)$ and makes the expressions under the integral sign bounded ($d\alpha / X(\alpha) = dx / 2 \cos {}^1/_2\alpha$).

Figures 2 and 3 depict the results of numerical computations for a shell when $R / h = 100$, $m = 2$, $1 - R / R_1 = 0.01$, $M = 20$. For $M = 40$ the other parameters change by less than 1%.

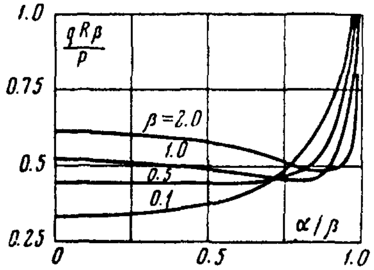

Fig. 2.

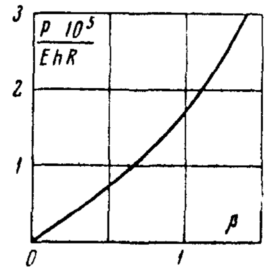

Fig. 3.

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